Synopsis on
Semidefinite Representation of Convex Sets

by

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The class of spectrahedra inherit many interesting properties of polyhedra and constitute the superclass of polyhedral sets. Thus, the class of spectrahedra plays an important role in applied mathematics and they generalize polyhedral sets. Spectrahedra and their projections, which are known as semidefinite representable sets, draw a considerable attention in combinatorial optimization [12],[15], polynomial optimization [6], [4], systems and control theory [1], [14], [13], [11] and convex algebraic geometry [5]. Semidefinite representable sets are the feasible regions of semidefinite programming problems. Semidefinite programming problems can be solved by efficient algorithms [2], [9], [10]. Thus, semidefinite programming [10], [2], [16], [9] is important in convex optimization [3].

A spectrahedron is a convex, semi-algebraic set which can be defined by linear matrix inequality, such as $S = \{x \in \mathbb{R}^n : A_0 + \sum_{i=1}^{n} A_i x_i \succeq 0\}$, where A_0, A_i's for $i = 1, 2, \ldots, n$ are all real symmetric matrices of same order. The matrix $A \succeq 0$ denotes that the matrix A is positive semidefinite. The inequality $A_0 + \sum_{i=1}^{n} A_i x_i \succeq 0$ is known as linear matrix inequality (LMI).

There exists convex set which is not spectrahedron but can be defined by a lifted LMI. The example is the TV screen: $\{(x, y) : x^4 + y^4 \leq 1\} \subseteq \mathbb{R}^2$. This example motivates to define another class of set, termed as lifted LMI set. This type of convex, semi-algebraic set in $\mathbb{R}^n$ which can be defined by a lifted LMI is called semidefinite representable set [2].Let S be a convex, semi-algebraic set such that

$$S = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m \text{ such that } A_0 + \sum_{i=1}^{n} A_i x_i + \sum_{j=1}^{m} B_j y_j \succeq 0\}, \tag{1}$$

then S is called semidefinite representable set where all the matrices A_0, A_i, B_j's for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ are real symmetric matrices. The representation in (1) is known as semidefinite representation of S. The set $S^{'} = \{(x, y) \in \mathbb{R}^{n+m} : A_0 + \sum_{i=1}^{n} A_i x_i + \sum_{j=1}^{m} B_j y_j \succeq 0\}$ projects down onto the space $\mathbb{R}^n$ and generates the convex set S where $S^{'}$ is a spectrahedron in $\mathbb{R}^{n+m}$.

We contribute several sufficient conditions for convex set to be semidefinite representable. We introduce new notions in semidefinite programming and we characterize those sets. The results are given below. We characterize convex cones considering sections. We also characterize cone(p, K).

We introduce a new notion in semidefinite programming, which is *compactly semidefinite representable set*. The definition is given below.

Definition 1 (Compactly semidefinite representable set). A subset K of $\mathbb{R}^n$ is said to be *compactly semidefinite representable set* provided its intersection with each compact semidefinite representable set in $\mathbb{R}^n$ is semidefinite representable.

Further we prove that *if a set is compactly semidefinite representable, then it is convex and semidefinite representable at all points.* We show that *if a set is closed, convex and semidefinite representable at all its points, then K is compactly semidefinite representable.*

We deal with the characterization of cone(p, K). We prove that *any convex cone is closed and semidefinite representable if and only if any hyperplane intersects the convex cone in a compactly semidefinite representable set and the cone contains its lineality space.* Further we develop sufficient condition for semidefinite representability of cone(p, K). We characterize sections of a closed convex cone $C \subseteq \mathbb{R}^n$ and we prove that *all r-sections of cone C are semidefinite representable if and only if $M \cap C$ is semidefinite representable for every $(r+1)$-flat M through the vertex of C.*

We investigated the sections of convex cone and derived that *the convex cone is semidefinite representable if its $(n-1)$-sections are compactly semidefinite representable.* It will be challenging if we can extend the **Section result** considering any generalized convex set. The sections and projections of convex set are connected to each other. So, using the **Section result** it will be challenging to derive the sufficient conditions for convex set using its projections. The main aim is to prove that a convex set is semidefinite representable or not, if its sections are semidefinite representable. Another interesting research scope is to prove that the convex set is semidefinite representable or not if its projections are semidefinite representable. So, the characterization of convex sets considering their sections and projections indeed gives challenging problems in semidefinite programming. More research is needed to prove the **section and projection result** for any convex set.

An active research aspect in convex optimization is about to find out the semidefinite representation of the convex sets. Till now only few methods have been proposed to construct the semidefinite representation. This section aims to contribute a new method to construct the explicit semidefinite representation of the convex body, say K in $\mathbb{R}^2$ if the projections of K on $\mathbb{R}$ are known. An algorithm has been proposed *to generate a sequence of core sets and the envelope sets of K such that the sequence of core sets and the sequence of envelope sets converge to the convex body K.* The basic idea is based on the idea of reconstruction of binary patterns [7] and on the idea of reconstruction of convex body from its projections [8].

The projection of a convex body $K \subseteq \mathbb{R}^2$ in the direction ν_i at the point u is denoted as $[P_{\nu_i}K](u)$. The **reconstruction problem** is to find out a compact convex set $K \subseteq \mathbb{R}^2$ such that the projections of K on $\mathbb{R}$ are known in finite number of directions with non-negative projection functions. The convex body K is said to be the solution of the problem. If C and $\text{conv}(E)$ on the set L' satisfy the following conditions:

(i) $C \subseteq \text{conv}(E) \subseteq \mathbb{R}^2$,

(ii) $(\text{conv}(E)\backslash C) \cap L' = \phi$,

(iii) $P_i C \leq p_i \leq P_i \,\text{conv}(E)$, for all $i = 1, 2, \ldots, M$,

we say $(C, \text{conv}(E))$ is **approximate solution** of the reconstruction problem. We develop an algorithm to generate approximate solution of the reconstruction problem. The output of the algorithm is the sequence of core sets, denoted by $\{C^{(k)}\}_{k=0}^{\infty}$ and the sequence of envelope sets, denoted by $\{\text{conv}(E^{(k)})\}_{k=0}^{\infty}$. We analyse the output and we show that the output constitutes the approximate solution to the reconstruction problem. We show that if the given input data is true, then the reconstruction algorithm terminates after finite number of steps. We characterize the output and we show that at k th step of the algorithm if $\text{conv}(E^{(k-1)}) \cap L' \neq \text{conv}(E^{(k)}) \cap L'$ holds, then the sequence of envelope sets and the sequence of core sets satisfy the relation: $\text{conv}\, E^{(k+1)} \subseteq \text{conv}\, E^{(k)}$ and $C^{(k)} \subseteq C^{(k+1)}$.

The core sets are polytopes. So, they are semidefinite representable. The envelope sets are convex and semi-algebraic sets in $\mathbb{R}^2$ by construction. So, they are semidefinite representable.

We prove that *the semidefinite representation of the convex body K is generated using relaxed LMI domination problem when K is a spectrahedron.* We calculate **the rate of convergence** of the core sets and the envelope sets. Further, we calculate **the stopping criteria** for the core sets and the envelope sets. We verify all the results using different examples.

This subsection deals with comparison of our construction method with other construction methods. We consider one example and we prove that our method gives approximate solution to the reconstruction problem but our method is very fast.

A new algorithm is introduced to construct semidefinite representation of compact convex set in $\mathbb{R}^2$ with non-empty interior. A method contributes a new construction method in semidefinite programming which can be applied for any convex body in $\mathbb{R}^2$. It will be challenging if we can generalize our construction method for any convex body in $\mathbb{R}^n$.

This section deals with introducing new notions in semidefinite programming and their characterization. Based on these characterizations, we develop sufficient conditions for semidefinite representability of convex sets.

Definition 2 (**Convex set semidefinite representable at infinity**). Let us assume that S is convex set in $\mathbb{R}^n$. The convex set S is called semidefinite representable at the point infinity if any semidefinite representable set or any translate of a closed halfspace in $\mathbb{R}^n$ intersecting the convex set S in a semidefinite representable set.

We prove that *a closed convex set $K \subseteq \mathbb{R}^n$ is semidefinite representable if and only if it is semidefinite representable at each point of K and infinity.*

Definition 3 (**Convex set semidefinite representable away from a point**). Let us consider a convex set K in $\mathbb{R}^n$ and p is any point in its closure. The set S semidefinite representable away from the point p is defined if and only if $S \cap K$ is semidefinite for any semidefinite representable set K in $\mathbb{R}^n \setminus \{p\}$.

We contribute that *a closed convex set $K \subseteq \mathbb{R}^n$ is semidefinite representable away from p if and only if K is semidefinite representable at each point of $K \cup \{\infty\} \setminus \{p\}$.*

We characterize compactly semidefinite representable sets considering their polar sets and various other ways. Based on this characterization, we give sufficient condition for any convex set to be semidefinite representable. We further investigate that *lower dimensional sets that are compactly semidefinite representable are projections of compactly semidefinite representable sets, and that projections of compactly semidefinite representable sets are also compactly semidefinite representable.*

We discuss the properties of the convex sets which are semidefinite representable away from the origin. We prove that *if a closed convex set $K \subseteq \mathbb{R}^n$ containing the origin, is semidefinite representable away from 0, then* $\text{cone}(p, K)$ *is semidefinite representable for every $x \in K \setminus \{0\}$.* We further contribute that *a closed convex set $K \subseteq \mathbb{R}^n$ is semidefinite representable if K and its polar K° both are semidefinite representable away from 0.*

We introduced a new notion in semidefinite programming, which is termed as *convex set being semidefinite representable at infinity.* It works as a natural tool to develop new sufficient condition for unbounded convex, semi-algebraic set to have semidefinite representation. It gives theoretical existence of semidefinite representation of convex sets. Another tool which is *convex set being semidefinite representable away from a point* has been contributed and is used to develop sufficient condition for semidefinite representability of a convex set. The *compactly semidefinite representable set* has been characterized in various ways. Another class of sets which are *semidefinite representable away from origin* have been characterized. We show that these characterizations are powerful tools in semidefinite programming for dealing with the analysis of semidefinite representability of unbounded convex semi-algebraic sets. This work can be extended to check the semidefinite representation of the convex sets under the contributed sufficient conditions.

We show that *any closed convex set can be approximated by a compactly semidefinite representable set*. If $K \subseteq \mathbb{R}^n$ is a closed convex set containing no lines, then there exists a compactly semidefinite representable set P such that $P \subseteq K \subseteq \cup_{x \in P} S(x, \mu x)$ where μ is a continuous function from K to $]0, \infty[$ and $S(x, \mu x)$ is the union of all open μx-neighbourhoods of the points x. We develop a method to construct a sequence of compactly semidefinite representable sets, say $\{P_i\}_{i=1}^{\infty}$ which give a more tighter approximation of closed convex set K at every step such that $P_1 \subseteq P_2 \subseteq \ldots \subseteq P_n \subseteq P_{n+1} \subseteq \ldots \subseteq K$. We show that *the sequence of compactly semidefinite representable sets strongly converge to the set* K. Further, we prove that *any closed convex set can be approximated uniformly by semidefinite representable set under some conditions*.

We presented few results on approximation of any closed convex set by compactly semidefinite representable set. Further we contributed a technique to approximate any closed convex set by semidefinite representable set under some condition. It will be challenging to develop technique to approximate any closed convex set by semidefinite representable set in general.

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